Case's method in higher dimensions with rotated reference frames

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Abstract. Case's method is generalized to higher spatial dimensions by reformulating the method of rotated reference frames.

1. Introduction

The method of solving the radiative transport equation with elementary solutions or singular eigenfunctions is called Case's method [1]. Although the method gives insight into the theoretical structure of the specific intensity, it works only for one spatial dimension [2]. On the other hand, another method called the method of rotated reference frames [3, 4] has been developed as an efficient numerical algorithm for higher spatial dimensions. In this paper, we extend Case's method to three dimensions by considering singular eigenfunctions in rotated reference frames. The calculation presented here turns out to be another formulation of the method of rotated reference frames. Thus, the generalization in this paper is not formal but numerically feasible.

The reminder of this paper is organized as follows. In section 2, we introduce the radiative transport equation. In section 3, we obtain normal modes. In section 4, we consider eigenvalues. Section 5 is devoted to the method of rotated reference frames. In section 6, we obtain the three-dimensional Green's function in an infinite medium. In section 7, we present the Green's function for isotropic scattering. Finally, we give summary in section 8. Polar and azimuthal angles in rotated reference frames are computed in Appendix A. The expansion coefficients in the method of rotated reference frames are calculated in Appendix B.

2. Radiative transport equation

Let $u(\mathbf{r},\hat{\mathbf{s}})$ be the specific intensity at position $\mathbf{r}\in\mathbb{R}^3$ in direction $\hat{\mathbf{s}}\in\mathbb{S}^2$. The (time-independent) radiative transport equation is given by

$$\hat{\mathbf{s}}\cdot\nabla u(\mathbf{r},\hat{\mathbf{s}})+u(\mathbf{r},\hat{\mathbf{s}})=c\int_{\mathbb{S}^2}f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')u(\mathbf{r},\hat{\mathbf{s}}')\,\mathrm{d}\hat{\mathbf{s}}'. \tag{1}$$

We assume $0<c<1$. Let $\mu=\cos\theta$ be the cosine of the polar angle of $\hat{\mathbf{s}}$ and φ be the azimuthal angle of $\hat{\mathbf{s}}$. Suppose the scattering phase function $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')$ can be modeled by a polynomial of spherical harmonics of order N.

$$f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')=\sum_{l=0}^{N}\sum_{m=-l}^{l}f_lY_{lm}(\hat{\mathbf{s}})Y_{lm}^*(\hat{\mathbf{s}}'). \tag{2}$$

Let $g=\int_{\mathbb{S}^2}(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\,\mathrm{d}\hat{\mathbf{s}}'$ and $\int_{\mathbb{S}^2}f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')\,\mathrm{d}\hat{\mathbf{s}}'=1$. We have $N=0$, $f_0=1$ in the case of isotropic scattering, $N=1$, $f_0=1$, $f_1=g$ in the case of linear scattering, and $N=\infty$, $f_l=g^l$ in the case of the Henyey-Greenstein model [5]. Let us express $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')$ as

$$f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')=\sum_{l=0}^{N}\sum_{m=-l}^{l}f_l\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\left(1-\mu^2\right)^{|m|/2}\left(1-{\mu'}^2\right)^{|m|/2}p_l^m(\mu)p_l^m(\mu')\mathrm{e}^{\mathrm{i}m(\varphi-\varphi')}. \tag{3}$$

Here the polynomials $p_l^m(\mu)$ are related to $P_l^m(\mu)$ as [6]

$$P_l^m(\mu)=(-1)^m\left(1-\mu^2\right)^{|m|/2}p_l^m(\mu). \tag{4}$$

They satisfy the following recurrence relations and orthogonality relations.

$$(l-m+1)p_{l+1}^m(\mu) = (2l+1)\mu p_l^m(\mu) - (l+m)p_{l-1}^m(\mu), \tag{5}$$

$$\int_{-1}^{1} p_l^m(\mu)p_{l'}^m(\mu)\,\mathrm{d}m(\mu) = \frac{2(l+m)!}{(2l+1)(l-m)!}\delta_{ll'}, \tag{6}$$

where we introduced

$$\mathrm{d}m(\mu) = \left(1-\mu^2\right)^{|m|}\,\mathrm{d}\mu. \tag{7}$$

3. Normal modes

We seek solutions of the form of plane-wave decomposition [7, 8]. We introduce $\nu \in \mathbb{R}$ and $\mathbf{q} \in \mathbb{R}^2$, and define vector $\mathbf{k} \in \mathbb{C}^3$ as

$$\mathbf{k} = \frac{1}{\nu}\hat{\mathbf{k}}, \quad \hat{\mathbf{k}} = \begin{pmatrix} -\mathrm{i}\nu\mathbf{q} \\ Q(\nu|\mathbf{q}|) \end{pmatrix}, \quad Q(\nu|\mathbf{q}|) = \sqrt{1+\nu^2|\mathbf{q}|^2}. \tag{8}$$

We emphasize that $\mathbf{k}$ and $\hat{\mathbf{k}}$ are functions of ν and $\mathbf{q}$. We assume the specific intensity of the form

$$u_\nu^m(\mathbf{r},\hat{\mathbf{s}};\mathbf{q}) = \Phi_\nu^m(\hat{\mathbf{s}};\hat{\mathbf{k}})\mathrm{e}^{-\mathbf{k}\cdot\mathbf{r}} = \phi^m(\nu,\mu(\hat{\mathbf{k}}))\left(1-\mu(\hat{\mathbf{k}})^2\right)^{|m|/2}\mathrm{e}^{\mathrm{i}m\varphi(\hat{\mathbf{k}})}\mathrm{e}^{-\mathbf{k}\cdot\mathbf{r}}, \tag{9}$$

where $\mu(\hat{\mathbf{k}})$ and $\varphi(\hat{\mathbf{k}})$ are the cosine of the polar angle of $\hat{\mathbf{s}}$ and the azimuthal angle of $\hat{\mathbf{s}}$, respectively, in the rotated reference frame whose z-axis coincides with the direction of $\hat{\mathbf{k}}$ (See Appendix A). We normalize ϕ^m as

$$\frac{1}{2\pi}\int_{\mathbb{S}^2}\phi^m(\nu,\mu(\hat{\mathbf{k}}))\left(1-\mu(\hat{\mathbf{k}})^2\right)^{|m|}\mathrm{d}\hat{\mathbf{s}} = \int_{-1}^{1}\phi^m(\nu,\mu)\,\mathrm{d}m(\mu) = 1. \tag{10}$$

By plugging the ansatz (9) into the radiative transport equation (1), we obtain

$$\begin{aligned}&\left(1-\frac{\mu(\hat{\mathbf{k}})}{\nu}\right)\phi^m(\nu,\mu(\hat{\mathbf{k}}))\left(1-\mu(\hat{\mathbf{k}})^2\right)^{|m|/2}\mathrm{e}^{\mathrm{i}m\varphi(\hat{\mathbf{k}})} = \\ &\qquad c\int_{\mathbb{S}^2} f\left(\hat{\mathbf{s}}(\hat{\mathbf{k}})\cdot\hat{\mathbf{s}}'(\hat{\mathbf{k}})\right)\phi^m(\nu,\mu'(\hat{\mathbf{k}}))\left(1-\mu'(\hat{\mathbf{k}})^2\right)^{|m|/2}\mathrm{e}^{\mathrm{i}m\varphi'(\hat{\mathbf{k}})}\,\mathrm{d}\hat{\mathbf{s}}',\end{aligned} \tag{11}$$

where we used $\hat{\mathbf{s}}\cdot\hat{\mathbf{k}} = \mu(\hat{\mathbf{k}})$ and expressed $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')$ in the rotated reference frame. The right-hand side is calculated as

$$\begin{aligned}\mathrm{RHS} = 2\pi c\Theta\left(N-|m|\right)&\left(1-\mu(\hat{\mathbf{k}})^2\right)^{|m|/2}\mathrm{e}^{\mathrm{i}m\varphi(\hat{\mathbf{k}})}\sum_{l'=|m|}^{N} f_{l'}\frac{2l'+1}{4\pi}\frac{(l'-m)!}{(l'+m)!} \\ &\times p_{l'}^m(\mu(\hat{\mathbf{k}}))\int_{-1}^{1}p_{l'}^m(\mu')\phi^m(\nu,\mu')\,\mathrm{d}m(\mu'),\end{aligned} \tag{12}$$

where the step function $\Theta(\cdot)$ is defined as $\Theta(x) = 1$ for $x \geq 0$ and $= 0$ for $x < 0$. Hence,

$$\left(\nu-\mu(\hat{\mathbf{k}})\right)\phi^m(\nu,\mu(\hat{\mathbf{k}})) = 2\pi c\nu\Theta\left(N-|m|\right)\sum_{l'=|m|}^{N} f_{l'}\frac{2l'+1}{4\pi}\frac{(l'-m)!}{(l'+m)!}p_{l'}^m(\mu(\hat{\mathbf{k}}))h_{l'}^m(\nu), \tag{13}$$

where we defined

$$h_l^m(\nu)=\int_{-1}^{1}\phi^m(\nu,\mu)p_l^m(\mu)\,\mathrm{d}m(\mu). \tag{14}$$

Since the right-hand side of (13) is zero for $|m|>N$ and then $\phi^m=0$, hereafter we suppose

$$0\le|m|\le N. \tag{15}$$

Let us define

$$\sigma_l=1-cf_l\Theta(N-l). \tag{16}$$

From (13), we obtain

$$\sigma_l\nu h_l^m(\nu)=\int_{-1}^{1}\mu\phi^m(\nu,\mu)p_l^m(\mu)\,\mathrm{d}m(\mu). \tag{17}$$

Thus the recurrence relation for $h_l^m(\nu)$ is obtained [9]:

$$\nu(2l+1)\sigma_l h_l^m(\nu)-(l-m+1)h_{l+1}^m(\nu)-(l+m)h_{l-1}^m(\nu)=0, \tag{18}$$

with $h_{|m|}^m(\nu)=(2m)!/2^m m!$ for $m\ge0$ and $h_{|m|}^m(\nu)=(-1)^m/2^{|m|}(|m|!)$ for $m<0$, and $h_{|m|+1}^{|m|}(\nu)=(2|m|+1)(\nu+\sigma_{|m|}-1)h_{|m|}^{|m|}(\nu)$. We also have

$$h_l^{-|m|}(\nu)=(-1)^{|m|}\frac{(l-|m|)!}{(l+|m|)!}h_l^{|m|}(\nu). \tag{19}$$

The functions $h_l^m(\nu)$ are computed using (18).

Let us define

$$\gamma^m(\nu,\mu(\hat{\mathbf{k}}))=\sum_{l'=|m|}^{N}(2l'+1)f_{l'}\frac{(l'-m)!}{(l'+m)!}p_{l'}^m(\mu(\hat{\mathbf{k}}))h_{l'}^m(\nu). \tag{20}$$

We note that $\gamma^{-m}(\nu,\mu(\hat{\mathbf{k}}))=\gamma^m(\nu,\mu(\hat{\mathbf{k}}))$. The function ϕ^m is obtained as

$$\phi^m(\nu,\mu(\hat{\mathbf{k}}))=\frac{c\nu}{2}\mathcal{P}\frac{\gamma^m(\nu,\mu(\hat{\mathbf{k}}))}{\nu-\mu(\hat{\mathbf{k}})}+\lambda^m(\nu)(1-\nu^2)^{-|m|}\delta(\nu-\mu(\hat{\mathbf{k}})). \tag{21}$$

4. Discrete eigenvalues and continuous spectrum

By integrating (21) with respect to $\hat{\mathbf{s}}$, we obtain

$$1=\frac{c\nu}{2}\,\mathcal{P}\!\int_{-1}^{1}\frac{\gamma^m(\nu,\mu)}{\nu-\mu}\,\mathrm{d}m(\mu)+\int_{-1}^{1}\lambda^m(\nu)\delta(\nu-\mu)\,\mathrm{d}\mu. \tag{22}$$

Let us define

$$\Lambda^m(z)=1-\frac{cz}{2}\int_{-1}^{1}\frac{\gamma^m(z,\mu)}{z-\mu}\,\mathrm{d}m(\mu), \tag{23}$$

where $z\in\mathbb{C}$. Eigenvalues $\nu\notin[-1,1]$ are solutions to

$$\Lambda^m(\nu)=0. \tag{24}$$

We write these discrete eigenvalues as $\pm\nu_j^m$ ($\nu_0^m > \nu_1^m > \cdots > \nu_{M-1}^m > 1$). Note that $\nu_j^{-m} = \nu_j^m$. The number of discrete eigenvalues M depends on $|m|$ and we have [10, 6] $M \le N - |m| + 1$.

For $\nu \in [-1, 1]$, we have

$$\lambda^m(\nu) = 1 - \frac{c\nu}{2}\,\mathcal{P}\int_{-1}^{1} \frac{\gamma^m(\nu,\mu)}{\nu-\mu}\,\mathrm{d}m(\mu). \tag{25}$$

Note that $\lambda^{-m}(\nu) = \lambda^m(\nu)$. This implies $\phi^{-m}(\nu, \mu(\hat{\mathbf{k}})) = \phi^m(\nu, \mu(\hat{\mathbf{k}}))$.

In two dimensions ($\mathbf{r} \in \mathbb{R}^2$, $\hat{\mathbf{s}} \in \mathbb{S}^1$), we have $\hat{\mathbf{s}} = (\cos\varphi, \sin\varphi)$ and $f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}') = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^N f_m \cos m(\varphi - \varphi')$. We can introduce elementary solutions as $u(\mathbf{r}, \hat{\mathbf{s}}) = \phi(\nu, \mu(\hat{\mathbf{k}}))\mathrm{e}^{-\mathbf{k}\cdot\mathbf{r}}$, where $\mu(\hat{\mathbf{k}}) = \cos(\varphi - \varphi_{\hat{\mathbf{k}}})$. Here $\mathbf{k} = \hat{\mathbf{k}}/\nu$ and $\hat{\mathbf{k}} = (\cos\varphi_{\hat{\mathbf{k}}}, \sin\varphi_{\hat{\mathbf{k}}}) = (-\mathrm{i}\nu q, Q(\nu q))$ with $q \in \mathbb{R}$. Let us define $h_m(\nu) = \int_0^{2\pi}\cos(m\varphi)\phi(\nu,\mu)\,\mathrm{d}\varphi$ and $\gamma(\nu, \mu(\hat{\mathbf{k}})) = \sum_{m=-N}^N f_m \cos\left(m\varphi(\hat{\mathbf{k}})\right) h_m(\nu)$. The singular eigenfunctions in two dimensions are obtained as

$$\phi(\nu, \mu(\hat{\mathbf{k}})) = \frac{c\nu}{2\pi}\mathcal{P}\frac{\gamma(\nu, \mu(\hat{\mathbf{k}}))}{\nu - \mu(\hat{\mathbf{k}})} + \lambda(\nu)\delta(\nu - \mu(\hat{\mathbf{k}})), \tag{26}$$

where $\lambda(\nu) = 1 - \frac{c\nu}{2\pi}\,\mathcal{P}\int_0^{2\pi}\frac{\gamma(\nu,\mu)}{\nu-\mu}\,\mathrm{d}\varphi$. Discrete eigenvalues are obtained as solutions to

$$\Lambda(z) = 1 - \frac{cz}{2\pi}\int_0^{2\pi}\frac{\gamma(z,\mu)}{z-\mu}\,\mathrm{d}\varphi = 0, \quad z \in \mathbb{C}. \tag{27}$$

In particular, for isotropic scattering $N = 0$, we have $\Lambda(z) = 1 - \frac{cz}{2\pi}\int_0^{2\pi}\frac{1}{z-\mu}\,\mathrm{d}\varphi = 1 - \frac{cz}{\sqrt{z^2-1}} = 0$. Therefore, we obtain $z = \pm 1/\sqrt{1-c^2} = \pm\nu_0$.

5. Method of rotated reference frames

Let us expand singular eigenfunctions with spherical harmonics. By introducing $c_l^m(\nu)$, we write

$$\Phi_\nu^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) = \sum_{l=|m|}^{\infty} c_l^m(\nu) Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}). \tag{28}$$

The calculation of the specific intensity by this expansion is called the method of rotated reference frames [3, 4]. From (11), we obtain

$$c_l^m(\nu) - \frac{1}{\nu}\sum_{l'=|m|}^{\infty}\left(\int_{\mathbb{S}^2}\mu Y_{l'm}(\hat{\mathbf{s}})Y_{lm}^*(\hat{\mathbf{s}})\,\mathrm{d}\hat{\mathbf{s}}\right)c_{l'}^m(\nu) = cf_l\Theta(N-l)c_l^m(\nu). \tag{29}$$

Hence we arrive at an eigenproblem:

$$B^m|\psi^m(\nu)\rangle = \nu|\psi^m(\nu)\rangle, \tag{30}$$

where

$$\begin{aligned} B_{ll'}^m &= \frac{1}{\sqrt{\sigma_l\sigma_{l'}}}\int_{\mathbb{S}^2}\mu Y_{l'm}(\hat{\mathbf{s}})Y_{lm}^*(\hat{\mathbf{s}})\,\mathrm{d}\hat{\mathbf{s}} \\ &= \sqrt{\frac{l^2 - m^2}{(4l^2-1)\sigma_l\sigma_{l-1}}}\delta_{l',l-1} + \sqrt{\frac{(l+1)^2 - m^2}{(4(l+1)^2-1)\sigma_{l+1}\sigma_l}}\delta_{l',l+1}, \end{aligned} \tag{31}$$

$$\langle l|\psi^m(\nu)\rangle = \frac{1}{\sqrt{Z^m(\nu)}}\sqrt{\sigma_l}c_l^m(\nu), \tag{32}$$

where the normalization $Z^m(\nu)$ will be determined below so that $\langle\psi^m(\nu)|\psi^m(\nu)\rangle = 1$ is satisfied. Note that ν depends on $|m|$. We obtain

$$\Phi_\nu^m(\hat{\mathbf{s}};\hat{\mathbf{k}}) = \sqrt{Z^m(\nu)}\sum_{l=|m|}^{\infty}\frac{\langle l|\psi^m(\nu)\rangle}{\sqrt{\sigma_l}}Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}}). \tag{33}$$

6. The Green's function

Let us consider the Green's function of the radiative transport equation in an infinite medium. The Green's function obeys

$$\hat{\mathbf{s}}\cdot\nabla G(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) + G(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = c\int_{\mathbb{S}^2} f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')G(\mathbf{r},\hat{\mathbf{s}}';\mathbf{r}_0,\hat{\mathbf{s}}_0)\,\mathrm{d}\hat{\mathbf{s}}' + \delta(\mathbf{r}-\mathbf{r}_0)\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}_0). \tag{34}$$

To proceed, we introduce $\tilde{\Phi}_\nu^m(\hat{\mathbf{s}};\hat{\mathbf{k}})$ such that

$$\int_{\mathbb{S}^2}\mu\Phi_\nu^m(\hat{\mathbf{s}};\hat{\mathbf{k}})\left[\tilde{\Phi}_{\nu'}^m(\hat{\mathbf{s}};\hat{\mathbf{k}}')\right]^*\mathrm{d}\hat{\mathbf{s}} = \delta_{\nu\nu'}, \tag{35}$$

where the Kronecker delta $\delta_{\nu\nu'}$ is understood as the Dirac delta $\delta(\nu-\nu')$ for the continuous spectrum. The function $\tilde{\Phi}_\nu^m(\hat{\mathbf{s}};\hat{\mathbf{k}})$ will be determined as we compute the Green's function.

We replace the source term in (34) by a jump condition and solve

$$\begin{cases} \hat{\mathbf{s}}\cdot\nabla G(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) + G(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = c\int_{\mathbb{S}^2} f(\hat{\mathbf{s}}\cdot\hat{\mathbf{s}}')G(\mathbf{r},\hat{\mathbf{s}}';\mathbf{r}_0,\hat{\mathbf{s}}_0)\,\mathrm{d}\hat{\mathbf{s}}', \\ G(\boldsymbol{\rho},z_0+0,\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) - G(\boldsymbol{\rho},z_0-0,\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = \dfrac{1}{\hat{\mathbf{s}}\cdot\hat{\mathbf{z}}}\delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)\delta\left(\hat{\mathbf{s}}-\hat{\mathbf{s}}_0\right), \end{cases} \tag{36}$$

with the boundary condition $\lim_{|\mathbf{r}|\to\infty}G(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = 0$ and $\mathbf{r} = (\boldsymbol{\rho},z)$ ($\boldsymbol{\rho}\in\mathbb{R}^2$, $z\in\mathbb{R}$). Let us expand the Green's function using normal modes (9).

$$\begin{cases} G(\boldsymbol{\rho},z,\hat{\mathbf{s}};\boldsymbol{\rho}_0,z_0,\hat{\mathbf{s}}_0) = \displaystyle\sum_{m=-N}^{N}\int_{\mathbb{R}^2}\left[\sum_{j=0}^{M-1}a_{j+}^m(\mathbf{q})u_{j+}^m(\mathbf{r},\hat{\mathbf{s}};\mathbf{q})\right. & \\ \left.+\displaystyle\int_0^1 A_\nu^m(\mathbf{q})u_\nu^m(\mathbf{r},\hat{\mathbf{s}};\mathbf{q})\,\mathrm{d}\nu\right]\dfrac{\mathrm{d}\mathbf{q}}{(2\pi)^2}, & z>z_0, \\ G(\boldsymbol{\rho},z,\hat{\mathbf{s}};\boldsymbol{\rho}_0,z_0,\hat{\mathbf{s}}_0) = -\displaystyle\sum_{m=-N}^{N}\int_{\mathbb{R}^2}\left[\sum_{j=0}^{M-1}a_{j-}^m(\mathbf{q})u_{j-}^m(\mathbf{r},\hat{\mathbf{s}};\mathbf{q})\right. & \\ \left.+\displaystyle\int_{-1}^0 A_\nu^m(\mathbf{q})u_\nu^m(\mathbf{r},\hat{\mathbf{s}};\mathbf{q})\,\mathrm{d}\nu\right]\dfrac{\mathrm{d}\mathbf{q}}{(2\pi)^2}, & z<z_0. \end{cases} \tag{37}$$

From the jump condition, coefficients $a_{j\pm}^m$ and A_ν^m are determined as

$$a_{j\pm}^m(\mathbf{q}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\boldsymbol{\rho}_0}\mathrm{e}^{\pm Q(\nu_j^m|\mathbf{q}|)z_0/\nu_j^m}\left[\tilde{\Phi}_{j\pm}^m(\hat{\mathbf{s}}_0;\hat{\mathbf{k}})\right]^*. \tag{38}$$

$$A_\nu^m(\mathbf{q}) = \mathrm{e}^{-\mathrm{i}\mathbf{q}\cdot\boldsymbol{\rho}_0}\mathrm{e}^{Q(\nu|\mathbf{q}|)z_0/\nu}\left[\tilde{\Phi}_\nu^m(\hat{\mathbf{s}}_0;\hat{\mathbf{k}})\right]^*. \tag{39}$$

Hence the Green's function is written as

$$G(\boldsymbol{\rho}, z, \hat{\mathbf{s}}; \boldsymbol{\rho}_0, z_0, \hat{\mathbf{s}}_0) = \frac{\pm 1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)} \sum_{m=-N}^{N} \Bigg\{$$

$$\sum_{j=0}^{M-1} \Phi_{j\pm}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\tilde{\Phi}_{j\pm}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{k}})\right]^* \mathrm{e}^{-Q(\nu_j^m|\mathbf{q}|)|z-z_0|/\nu_j^m}$$

$$+ \int_0^1 \Phi_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\tilde{\Phi}_{\pm\nu}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{k}})\right]^* \mathrm{e}^{-Q(\nu|\mathbf{q}|)|z-z_0|/\nu} \, \mathrm{d}\nu \Bigg\} \, \mathrm{d}\mathbf{q}, \tag{40}$$

where upper signs are chosen for $z > z_0$ and lower signs are chosen for $z < z_0$.

Noting the relation (33), the Green's function obtained with the method of rotated reference frames [4] is expressed as

$$G(\mathbf{r}, \hat{\mathbf{s}}; \mathbf{r}_0, \hat{\mathbf{s}}_0) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)} \sum_{\nu>0} \sum_{m=-N}^{N} \frac{1}{\nu Q(\nu|\mathbf{q}|) Z^m(\nu)}$$

$$\times \, \Phi_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\Phi_{\pm\nu}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{k}})\right]^* \mathrm{e}^{-Q(\nu|\mathbf{q}|)|z-z_0|/\nu} \, \mathrm{d}\mathbf{q}. \tag{41}$$

By comparing (40) and (41), we obtain

$$\tilde{\Phi}_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) = \Phi_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\pm\nu Q(\nu|\mathbf{q}|) Z^m(\nu)\right]^{-1}. \tag{42}$$

To determine $Z^m(\nu)$, we consider the one-dimensional case. By integrate the Green's function over $\boldsymbol{\rho}_0$, we obtain

$$G(z, \hat{\mathbf{s}}; z_0, \hat{\mathbf{s}}_0) = \pm \sum_{m=-N}^{N} \Bigg\{$$

$$\sum_{j=0}^{M-1} \Phi_{j\pm}^m(\hat{\mathbf{s}}; \hat{\mathbf{z}}) \left[\tilde{\Phi}_{j\pm}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{z}})\right]^* \mathrm{e}^{-|z-z_0|/\nu_j^m} + \int_0^1 \Phi_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{z}}) \left[\tilde{\Phi}_{\pm\nu}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{z}})\right]^* \mathrm{e}^{-|z-z_0|/\nu} \, \mathrm{d}\nu \Bigg\}. \tag{43}$$

On the other hand, the one-dimensional Green's function is given by [10, 6]

$$G(z, \hat{\mathbf{s}}; z_0, \hat{\mathbf{s}}_0) = \frac{1}{2\pi} \sum_{m=-N}^{N} \Bigg\{$$

$$\sum_{j=0}^{M-1} \frac{1}{\mathcal{N}_j^m} \phi^m(\pm\nu_j^m, \mu) \phi^m(\pm\nu_j^m, \mu_0) \left(1-\mu^2\right)^{|m|} \mathrm{e}^{-|z-z_0|/\nu_j^m}$$

$$+ \int_0^1 \frac{1}{\mathcal{N}^m(\nu)} \phi^m(\pm\nu, \mu) \phi^m(\pm\nu, \mu_0) \left(1-\mu^2\right)^{|m|} \mathrm{e}^{-|z-z_0|/\nu} \, \mathrm{d}\nu \Bigg\}, \tag{44}$$

where

$$\mathcal{N}_j^m = \frac{1}{2} (\nu_j^m)^2 \gamma(\nu_j^m, \nu_j^m) \frac{\mathrm{d}\Lambda^m(z)}{\mathrm{d}z} \bigg|_{z=\nu_j^m}, \tag{45}$$

$$\mathcal{N}^m(\nu) = \nu \Lambda^{m+}(\nu) \Lambda^{m-}(\nu) \left(1-\nu^2\right)^{-|m|}. \tag{46}$$

Here $\Lambda^{m\pm}(\nu) = \lim_{\epsilon\to 0+} \Lambda^m(\nu \pm \mathrm{i}\epsilon)$. By comparing (43) and (44), we obtain

$$\nu_j^m Z^m(\nu_j^m) = 2\pi \mathcal{N}_j^m, \quad \nu Z^m(\nu) = 2\pi \mathcal{N}(\nu). \tag{47}$$

Finally we obtain the Green's function as

$$G(\boldsymbol{\rho}, z, \hat{\mathbf{s}}; \boldsymbol{\rho}_0, z_0, \hat{\mathbf{s}}_0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)} \sum_{m=-N}^{N} \Bigg\{$$
$$\sum_{j=0}^{M-1} \frac{1}{Q(\nu_j^m|\mathbf{q}|)\mathcal{N}_j^m} \Phi_{j\pm}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\Phi_{j\pm}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{k}})\right]^* \mathrm{e}^{-Q(\nu_j^m|\mathbf{q}|)|z-z_0|/\nu_j^m}$$
$$+ \int_0^1 \frac{1}{Q(\nu|\mathbf{q}|)\mathcal{N}(\nu)} \Phi_{\pm\nu}^m(\hat{\mathbf{s}}; \hat{\mathbf{k}}) \left[\Phi_{\pm\nu}^m(\hat{\mathbf{s}}_0; \hat{\mathbf{k}})\right]^* \mathrm{e}^{-Q(\nu|\mathbf{q}|)|z-z_0|/\nu}\, \mathrm{d}\nu \Bigg\}\, \mathrm{d}\mathbf{q}. \quad (48)$$

7. Isotropic scattering

For the isotropic scattering $N = 0$, we have

$$\Phi_\nu^0(\hat{\mathbf{s}}; \hat{\mathbf{k}}) = \frac{c\nu}{2} \mathcal{P} \frac{1}{\nu - \mu(\hat{\mathbf{k}})} + \lambda^0(\nu)\delta(\nu - \mu(\hat{\mathbf{k}})) = \phi(\nu, \mu(\hat{\mathbf{k}})), \quad (49)$$

where

$$\lambda^0(\nu) = 1 - \frac{c\nu}{2}\, \mathcal{P}\int_{-1}^{1} \frac{1}{\nu - \mu}\, \mathrm{d}\mu = 1 - c\nu \tanh^{-1}\nu. \quad (50)$$

Here $\mu(\hat{\mathbf{k}})$ is given in Appendix A. In this case $M = 1$ and the discrete eigenvalues $\pm\nu_0^0 = \pm\nu_0$ are solutions to

$$\Lambda^0(z) = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{1}{z - \mu}\, \mathrm{d}\mu = 1 - cz \tanh^{-1}\frac{1}{z} = 0. \quad (51)$$

The Green's function is obtained as

$$G(\boldsymbol{\rho}, z, \hat{\mathbf{s}}; \boldsymbol{\rho}_0, z_0, \hat{\mathbf{s}}_0) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)} \Bigg\{$$
$$\frac{1}{Q(\nu_0|\mathbf{q}|)\mathcal{N}_0} \phi\left(\pm\nu_0, \mu(\hat{\mathbf{k}})\right) \phi^*\left(\pm\nu_0, \mu_0(\hat{\mathbf{k}})\right) \mathrm{e}^{-Q(\nu_0|\mathbf{q}|)|z-z_0|/\nu_0}$$
$$+ \int_0^1 \frac{1}{Q(\nu|\mathbf{q}|)\mathcal{N}(\nu)} \phi\left(\pm\nu, \mu(\hat{\mathbf{k}})\right) \phi^*\left(\pm\nu, \mu_0(\hat{\mathbf{k}})\right) \mathrm{e}^{-Q(\nu|\mathbf{q}|)|z-z_0|/\nu}\, \mathrm{d}\nu \Bigg\}\, \mathrm{d}\mathbf{q}. \quad (52)$$

8. Summary

We have constructed a three-dimensional version of Case's method of solving the radiative transport equation. Extending Case's method to higher spatial dimensions was considered in 1960's shortly after Case's paper [1]. However the generalization is only formal and cannot be used for numerical calculation [2, 11, 7]. In this paper, the singular eigenfunctions are generalized using rotated reference frames and they are labeled by Case's discrete eigenvalues and continuous spectrum. The method developed in this paper is equivalent to the method of rotated reference frames. Therefore, the present method can also be considered a numerical method. Indeed, the three-dimensional radiative transport equation with anisotropic scattering in an infinite medium [3, 4] and

in the slab geometry [12] were numerically calculated. The radiative transport equation is also solved in two dimensions [13, 14, 15, 16, 17]. The use of the method of rotated reference frames or the Green's function (48) for inverse problems is also proposed [18].

Appendix A. Polar and azimuthal angles in rotated reference frames

Let θ and φ be the polar and azimuthal angles of $\hat{\mathbf{s}}$ in the laboratory frame. Let $\varphi_{\hat{\mathbf{k}}}$ and $\theta_{\hat{\mathbf{k}}}$ be the polar and azimuthal angles of $\hat{\mathbf{k}}$ in the laboratory frame. For $\hat{\mathbf{k}} = (-\mathrm{i}\nu\mathbf{q},\, Q(\nu|\mathbf{q}|))$, we obtain

$$\cos\theta_{\hat{\mathbf{k}}} = \hat{\mathbf{k}}\cdot\hat{\mathbf{z}} = Q(\nu|\mathbf{q}|), \quad \sin\theta_{\hat{\mathbf{k}}} = \sqrt{1-\cos^2\theta_{\hat{\mathbf{k}}}} = \mathrm{i}|\nu\mathbf{q}|, \tag{A.1}$$

and

$$\varphi_{\hat{\mathbf{k}}} = \begin{cases} \varphi_{\mathbf{q}} + \pi & \nu > 0, \\ \varphi_{\mathbf{q}} & \nu < 0, \end{cases} \tag{A.2}$$

where $\varphi_{\mathbf{q}}$ is the angle of $\mathbf{q}$. Therefore, we have

$$\mu(\hat{\mathbf{k}}) = \hat{\mathbf{s}}\cdot\hat{\mathbf{k}} = -\mathrm{i}\nu|\mathbf{q}|\sin\theta\cos(\varphi-\varphi_{\mathbf{q}}) + Q(\nu|\mathbf{q}|)\cos\theta. \tag{A.3}$$

In general, we can rotate functions as follows. Let us introduce rotated spherical harmonics $Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}})$ [3]:

$$Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}}) = \mathcal{D}(\hat{\mathbf{k}})Y_{lm}(\hat{\mathbf{s}}) = \sum_{m'=-l}^{l} D^l_{m'm}(\varphi_{\hat{\mathbf{k}}},\theta_{\hat{\mathbf{k}}},0)Y_{lm'}(\hat{\mathbf{s}}), \tag{A.4}$$

where $D^l_{m'm}(\varphi_{\hat{\mathbf{k}}},\theta_{\hat{\mathbf{k}}},0) = \mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}d^l_{m'm}(\theta_{\hat{\mathbf{k}}})$. Here $d^l_{m'm}$ are the Wigner d-matrices [19]. That is, $Y_{lm}(\hat{\mathbf{s}},\hat{\mathbf{k}})$ are spherical harmonics defined in a rotated reference frame whose z-axis coincides with the direction of the unit vector $\hat{\mathbf{k}}$. We have $Y_{lm}(\hat{\mathbf{s}}) = Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{z}})$. We write analytically continued Wigner's d-matrices as

$$d^l_{m'm}(\theta_{\hat{\mathbf{k}}}) = d^l_{m'm}[\mathrm{i}\tau(\nu|\mathbf{q}|)]. \tag{A.5}$$

First a few matrices are obtained as

$$d^0_{00} = 1, \quad d^1_{00} = \sqrt{1+x^2}, \quad d^1_{01} = \frac{\mathrm{i}}{\sqrt{2}}|x|, \quad d^1_{1\pm1} = \frac{1\pm\sqrt{1+x^2}}{2}. \tag{A.6}$$

We note that $d^l_{mm'} = (-1)^{m+m'}d^l_{-m-m'} = (-1)^{m+m'}d^l_{m'm}$. All $d^l_{m'm}[\mathrm{i}\tau(\nu|\mathbf{q}|)]$ are computed using the recurrence relations [12]. We obtain

$$\mathrm{e}^{\mathrm{i}m\varphi(\hat{\mathbf{k}})} = \left(1-\mu(\hat{\mathbf{k}})^2\right)^{-m/2}\frac{(-1)^m\sqrt{4\pi(2m+1)!}}{(2m+1)!!}\sum_{m'=-m}^{m}\mathrm{e}^{-\mathrm{i}m'\varphi_{\hat{\mathbf{k}}}}d^m_{m'm}(\theta_{\hat{\mathbf{k}}})Y_{mm'}(\hat{\mathbf{s}}), \tag{A.7}$$

where θ satisfies $\cos\theta = \mu$ with μ in (9).

Appendix B. Expansion coefficients

Here we calculate $c_l^m(\nu)$. We have

$$c_l^m(\nu) = \int_{\mathbb{S}^2} \left[\frac{c\nu}{2} \mathcal{P} \frac{\gamma^m(\nu,\mu)}{\nu-\mu} + \lambda^m(\nu) \left(1-\nu^2\right)^{-|m|} \delta(\nu-\mu) \right] \times \left(1-\mu^2\right)^{|m|/2} \mathrm{e}^{\mathrm{i}m\varphi} Y_{lm}^*(\hat{\mathbf{s}})\,\mathrm{d}\hat{\mathbf{s}}. \tag{B.1}$$

Hence,

$$c_l^m(\nu) = 2\pi \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \Bigg[\frac{c\nu}{2} \sum_{l''=|m|}^{N} f_{l''}(2l''+1) \frac{(l''-m)!}{(l''+m)!} h_{l''}^m(\nu) \times (-1)^m\, \mathcal{P}\!\int_{-1}^{1} \frac{P_{l''}^m(\mu) P_l^m(\mu)}{\nu-\mu}\,\mathrm{d}\mu + \lambda^m(\nu) \left(1-\nu^2\right)^{-|m|/2} P_l^m(\nu) \int_{-1}^{1} \delta(\nu-\mu)\,\mathrm{d}\mu \Bigg]. \tag{B.2}$$

Note that $c_l^m(-\nu) = (-1)^{l+m} c_l^m(\nu)$ because $P_l^m(-\nu) = (-1)^{l+m} P_l^m(\nu)$. Therefore, we obtain for $\nu \notin [-1,1]$

$$c_l^m(\nu) = 2\pi \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \frac{c\nu}{2} \sum_{l''=|m|}^{N} f_{l''}(2l''+1) \frac{(l''-m)!}{(l''+m)!} h_{l''}^m(\nu) \times (-1)^m 2 Q_{\max(l,l'')}^m(\nu) P_{\min(l,l'')}^m(\nu), \tag{B.3}$$

where $Q_{\max(l,l'')}^m(\nu)$ and $P_{\min(l,l'')}^m(\nu)$ have a branch cut from $-\infty$ to 1 [20], and for $\nu \in [-1,1]$

$$c_l^m(\nu) = 2\pi \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \Bigg[\frac{c\nu(-1)^m}{2} \sum_{l''=|m|}^{N} f_{l''}(2l''+1) \frac{(l''-m)!}{(l''+m)!} h_{l''}^m(\nu) \times \left(-\mathrm{i}\pi P_{l''}^m(\nu) P_l^m(\nu) - \int_0^{\pi} \frac{P_{l''}^m(\mathrm{e}^{\mathrm{i}\theta}) P_l^m(\mathrm{e}^{\mathrm{i}\theta})}{\nu - \mathrm{e}^{\mathrm{i}\theta}} \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \right) + \lambda^m(\nu)(1-\nu^2)^{-|m|/2} P_l^m(\nu) \Bigg]. \tag{B.4}$$

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